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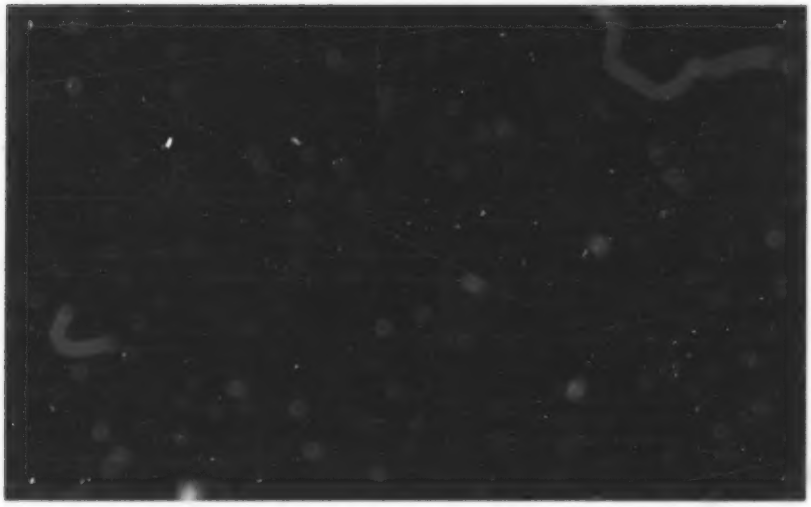

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INCREASES IN RISK AND LINEAR PAYOFFS*

by

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ABSTRACT

This paper is concerned with the effect of increases in risk on optimal decision variables for the class of linear payoffs. We show that for this class of payoffs one can extend the class of admissible increases in risk and obtain the desirable comparative statics properties. We propose the definition of a "relative weak increase in risk" and apply it to the case of the competitive firm with constant marginal costs and to the standard portfolio problem. Another application is the coinsurance problem.

Key words : increases in risk, comparative statics analysis, relatively weak increases in risk, strong increases in risk, linear payoffs, portfolio problem, competitive firm under uncertainty, coinsurance problem.

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RÉSUMÉ

Cette étude s'intéresse aux effets des accroissements de risque sur les décisions optimales des agents économiques ayant des payoffs linéaires. Nous montrons que, pour cette classe de payoffs, il est possible d'utiliser une définition d'accroissement de risque générale (accroissement de risque relativement faible) pour obtenir les résultats de statique comparative désirés. Nous appliquons notre définition d'accroissement de risque au problème de l'entreprise concurrentielle caractérisée par des coûts marginaux constants et au problème du portefeuille. Une autre application est le modèle de coassurance.

Mots-clés : accroissements de risque, analyse de statique comparative, accroissements de risque relativement faibles, accroissements de risque forts, payoff linéaire, problème de portefeuille, entreprise concurrentielle en incertitude, coassurance.

1. INTRODUCTION

In two important contributions, Meyer and Ormiston [1983, 1985] proposed the definition of a "strong increase in risk" in order to obtain intuitively appealing comparative statics results for a wide class of economic problems. Their definition of a strong increase in risk is a particular case of a mean preserving spread introduced in the literature by Rothschild and Stiglitz [1970, 1971]. It was motivated by the fact that a mean preserving spread is too broad to yield general comparative statics results for all risk-averse agents.[1]

As mentioned by Rothschild and Stiglitz themselves, in general one must restrict either the set of utility functions or the set of mean preserving increases in risk to sign the comparative statics results in accordance with intuition. For example, they obtain that a mean preserving spread reduces the proportion of wealth invested in the risky asset by all risk averse agents when the risky asset is characterized by a bet both before and after the change in risk or when the agent has a quadratic utility function.

Meyer and Ormiston [1985] show that their definition of a strong increase in risk is useful to obtain intuitive results for economic applications such as a competitive firm under price uncertainty, a cooperative firm and the problem of hiring workers. However, they were not able to apply their basic Theorem 1 to the standard portfolio problem. The main reason for this negative result is that one of the conditions for Theorem 1 to work, namely that the payoff function has to be strictly concave in the decision variable, is not satisfied in this portfolio problem.

More recently, Black and Bulkley [1989] introduced the definition of a relatively strong increase in risk. Their purpose was to extend the class of admissible changes in risk that yield the desired comparative statics results. However, as they put the same restrictions on the payoff function as Meyer and Ormiston, they are also unable to meaningfully deal with the standard portfolio problem, as well as other economic problems with linear payoffs.

In this paper we show that for the class of linear payoffs one can further extend the class of admissible increases in risk and still obtain the desirable comparative statics properties. Since this class of increases in risk includes as special cases the relatively strong increases in risk and, by transitivity, the strong increases in risk, we propose to name it the class of "relatively weak increases in risk". We apply the definition to the case of the competitive firm with constant marginal costs and to the standard portfolio problem.

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[1] See Sandmo [1971], Eeckhoudt and Hansen [1980, 1983], Meyer [1989] and Meyer and Ormiston [1989] for other definitions of increases in risk.

2. THE MODEL AND MAIN RESULTS

Following Meyer and Ormiston [1985], we use the general economic model introduced by Kraus [1979] and Katz [1981] where the agent maximizes his expected utility by choosing a value of the decision variable b for a given distribution of the random variable x. Formally, the agent maximizes EU[Z(b, x)] over scalar b where Z is a real valued function and U is the von Neuman-Morgenstern utility function.

We assume that $U'(Z) \geq 0$ and $U''(Z) < 0$ which implies that the decision maker is risk averse. Since we assume here the payoffs to be linear, they may (but need not) be expressed as

$$Z(x, b) = b(x - c) + Z_0 , \qquad (1)$$

where Z_0 and c are exogenous constants. Z_0 can be interpreted as an initial endowment or a fixed cost, while c may represent either a marginal cost or a sure interest rate.[2] Obviously here $Z_x \geq 0$, when $b \geq 0$, $Z_{bx} > 0$, Z_{bb} and Z_{xx} are equal to zero. In some applications (portfolio and insurance problems for example) Z_x can be negative but this characteristic does not affect the results when appropriate modifications are made. In order to simplify the presentation of the model we shall restrict its interpretation in this section to the case where $Z_x > 0$. In the discussion of the portfolio problem we consider all cases.

To discuss in detail the effect of an increase in the riskiness of a random variable, let us denote F(x) as the initial distribution function of x and G(x) the final distribution. Each distribution has a probability density function f(x) and g(x). We use Meyer and Ormiston's assumption concerning the supports of each distribution, that is, F(x) is contained in the bounded interval $[x_2, x_3]$, while G(x) is also contained in a bounded interval $[x_1, x_4]$ where $x_1 \leq x_2 \leq x_3 \leq x_4$.

The first order condition that determines the optimal value of b in the initial risky situation is

$$\int_{x_2}^{x_3} U'(Z(x,b))\, Z_b\, dF(x) = 0$$

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[2] In the coinsurance example, the linear payoff is equal to $W_0+\alpha(x-\lambda\mu)-x$ where α is the coinsurance rate, $\alpha\lambda\mu$ the insurance premium, W_0 the initial wealth and x the amount of loss. It can be rewritten as $W_0-\lambda\mu-(1-\alpha)(x-\lambda\mu)$ which is (1) with $Z_0=W_0-\lambda\mu$, $b=-(1-\alpha)$ and $c=\lambda\mu$.

and, under risk aversion, this condition is necessary and sufficient for a global optimum corresponding to the value b^*.

It is worth noticing that the linear payoff structure enables us to use a definition of a mean preserving increase in risk that is less restrictive than a relatively strong increase in risk. Formally, our definition of a relatively weak increase in risk is :

$$(a) \quad \int_{x_1}^{x_4} \big(G(x) - F(x)\big)\, dx = 0$$

$$(b) \quad \int_{x_1}^{y} \big(G(x) - F(x)\big) dx \geq 0 \text{ for all } y \leq x_4 \qquad (2)$$

(c) *For all points in an interval* $[x_5, x_6]$ *with* $x_2 \leq x_5$ *and* $x_6 \leq x_3$, $\dfrac{f(x)}{g(x)} \geq 1$, *and for all points outside that interval,* $\dfrac{f(x)}{g(x)} \leq 1$.

Conditions (a) and (b) are well-known for a mean preserving spread. The idea behind condition (c) is that after an increase is risk, the likelihood of extreme values should become higher in the new density, while that of interior values should become lower. This idea is formalized by reference to the likelihood ratio $\dfrac{f(x)}{g(x)}$.

This condition (c) restricts the set of admissible mean preserving spreads. However, it is important to note that with a relatively weak increase in risk, we do not need conditions (c) and (d) of Black and Bulkley about the behaviour of likelihood ratios while our above condition (c) corresponds to their condition (b).[3]

We are now in a position to state our major result for marginal increases in risk.

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[3]Technically and contrary to a claim in Black and Bulkley [1989], (a) and (c) in (2) are not sufficient for G(x) to represent a mean preserving spread in the Rothschild and Stiglitz sense. They need a condition like condition (b) in (2).

THEOREM : If b^* and b^{**} maximize E[U(Z(x, b))] under F(x) and G(x) respectively, sufficient conditions for $b^{**} < b^*$ for all risk averse decision makers are :

(i) G(x) represents a relatively weak increase in risk of F(x);
(ii) Z(x, b) is a linear payoff.

PROOF :

The proof starts from the first order condition for the initial distribution function F(x)

$$E\left[U'(Z(x,b)) \cdot Z_b\right] = 0 \ . \tag{3}$$

As mentioned above, under risk aversion (3) is necessary and sufficient to yield a unique interior solution b^*. Under the new distribution function G(x), one has to show that (3) evaluated at b^* is negative. Consequently, we have to show that

$$\int_{x_1}^{x_8} U'(Z(x,b^*))\ Z_b\ s(x)\ dx < 0 \tag{4}$$

where $s(x) \equiv g(x) - f(x)$.

Three cases can emerge depending of the relative position of c with respect of x_5 and x_6 : (i) $x_5 \le c \le x_6$, (ii) $x_5 \ge c$, and (iii) $c \ge x_6$, where c is defined in (1). The proof will be given here only for case (i) and it can be repeated with appropriate modifications for the two other cases.

For case (i), we can rewrite (4) as

$$\begin{aligned} &\int_{x_1}^{x_5} U'(Z(x,b^*))\ Z_b\ s(x)\ dx + \int_{x_5}^{c} U'(Z(x,b^*))\ Z_b\ s(x)\ dx \\ &+\int_{c}^{x_6} U'(Z(x,b^*))\ Z_b\ s(x)\ dx + \int_{x_6}^{x_8} U'(Z(x,b^*))\ Z_b\ s(x)\ dx < 0 \ . \end{aligned} \tag{5}$$

From the definition of a relatively weak increase in risk, the integrands in (5) alternate in sign starting with a negative one. Because $U'(Z(x, b^*))$ is decreasing in x, the left-hand side of (5) is smaller than

$$U'\left(Z(x_5, b^*)\right)\left[\int_{x_1}^{x_5} Z_b\, s(x)\, dx + \int_{x_5}^{c} Z_b\, s(x)\, dx\right] + U'\left(Z(x_6, b^*)\right)\left[\int_{c}^{x_6} Z_b\, s(x)\, dx + \int_{x_6}^{x_4} Z_b\, s(x)\, dx\right] \tag{6}$$

where $Z(x_i, b^*)(i = 5,6)$ represents the payoffs evaluated at x_i and b^*.

The sum of the two expressions in brackets is clearly equal to zero for any payoff linear in x and b. Indeed using (1), one obtains $Z_b = (x-c)$ so that the sum of the brackets can also be written as

$$\int_{x_1}^{x_4} (x-c)\, s(x)\, dx = \int_{x_1}^{x_4} x\, s(x)\, dx - c\int_{x_1}^{x_4} s(x)\, dx$$

which is equal to zero since the two terms on the right-hand side are themselves equal to zero by definition of a mean preserving increase in risk.

Since $U'(Z(x_5, b^*)) > U'(Z(x_6, b^*))$ under risk aversion, to complete the proof we now show that the first bracket in (6) is negative. Integrating by parts and using (1), one obtains

$$\int_{x_1}^{c} (x-c)\, s(x)\, dx = (x-c)\, S(x) \Big|_{x_1}^{c} - \int_{x_1}^{c} S(x)\, dx$$

$$= -\int_{x_1}^{c} S(x)\, dx$$

which is negative by condition (b) in the definition of the relatively weak increase in risk.

Q.E.D.

A direct corollary can be stated as follows :

COROLLARY : When $Z_x \leq 0$, if b^* and b^{**} maximize $E[U(Z(x,b))]$ under $F(x)$ and $G(x)$ respectively, conditions (i) and (ii) of the Theorem are sufficient for $b^* \leq b^{**}$ for all risk averse decision makers.

Let us now discuss the case of a global increase in risk. If $F(x)$ is a degenerate distribution with probability equal to one at x_0, it is clear that either corner solutions or indeterminate solutions would emerge since $Z(b, x_0)$ is linear in b. We can easily show that the transition from certainty to uncertainty will necessarily induce a non positive effect (in absolute value) on the optimal solutions under certainty for all risk-averse individuals. Examples are given in the next section.

3. EXAMPLES AND CONCLUSION

Let us now present two well-known applications of linear payoffs.

3.1 Standard Portfolio Problem

The payoff of the standard portfolio problem can be written as in Meyer and Ormiston (1985)

$$Z(x, b) = Z_0 (1+dx)$$

where d is the fraction of Z_0 allocated to the risky asset and where the sure interest rate is null. This payoff is equivalent to (1) when $b \equiv dZ_0$ and $c = 0$.

Considering a marginal change in risk, one can solve the usual portfolio problem under $F(x)$ where an optimal solution exists for all risk-averse individuals. Then, by the Theorem

and the Corollary, we can show that all risk-averse investors will hold a smaller proportion (in absolute value) of their wealth ($|b^{\cdot\cdot}| < |b^{\cdot}|$) in the risky asset under G(x) if G(x) represents a relatively weak increase in risk.

Let us now consider a global increase in risk. Three cases are possible under certainty when borrowing and short selling are allowed : 1) $x^0 > 0$ and $b^{\cdot} = +\infty$; 2) $x^0 < 0$ and $b^{\cdot} = -\infty$ and 3) $x^0 = 0$ and $b^{\cdot}$ is indeterminate. If x now becomes a random variable with an expected value equal to x^0, one obtains for all risk-averse individuals : 1) $b^{\cdot\cdot} \leq +\infty$; 2) $b^{\cdot\cdot} \geq -\infty$ and 3) $b^{\cdot\cdot} = 0$; as indicated at the end of the previous section, $|b^{\cdot\cdot}| \leq |b^{\cdot}|$ which means that all risk averse investors take a less risky position in all three cases.

3.2 The Competitive Firm with Constant Marginal Cost

For the competitive firm, the linear profit function is $Z(x, b) = b(x\text{-}c) + Z_0$, where x is the output price, c the marginal cost, Z_0 the fixed cost and b the level of output.

This example is slightly different from the preceding one since b is usually restricted to non-negative values. One can replicate the analysis developed above and obtain clearcut comparative statics results both for a global and for a marginal increase in risk.[4]

The interesting feature of our approach is that we can analyze the case of constant marginal costs and, at the same time, consider a broader class of mean preserving increases in risk than the strong or relatively strong ones. As already noted, both Meyer and Ormiston and Black and Bulkley require the Z function to be strictly concave in b to apply their basic theorems.

3.3 Conclusion

While linear payoffs turn out to be a special case in the set of decision problems under uncertainty, they appear to prevail in many economic environments. We have shown in this paper that for this class of problems one can obtain appealing comparative statics results with relatively minor restrictions on the set of mean preserving increases in risk.

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[4]Under certainty, three cases are again possible : 1) $(x^0\text{-}c) > 0$ and $b^{\cdot} = +\infty$; 2) $(x^0\text{-}c) < 0$ and $b^{\cdot} = 0$; and 3) $(x^0\text{-}c) = 0$ and $b^{\cdot}$ is indeterminate. If the output price becomes a random variable with a mean equal to x^0, one obtains for all risk-averse entrepreneurs : 1) $b^{\cdot\cdot} \leq +\infty$; 2) $b^{\cdot\cdot} = 0$; and 3) $b^{\cdot\cdot} = 0$; again $b^{\cdot\cdot} \leq b^{\cdot}$ in all three cases. For marginal increases in risk, the Theorem and the Corollary apply directly.

This general conclusion was applied to both the standard portfolio model and the optimal behaviour of a competitive firm with constant marginal costs. Another application is the coinsurance problem (see Alarie, Dionne and Eeckhoudt for more details).

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